Black Holes on Thin 3-branes of Codimension-2 and their Extension into the Bulk

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Abstract

We discuss black hole solutions in six-dimensional gravity with a Gauss-Bonnet term in the bulk and an induced gravity term on a thin 3-brane of codimension-2. We show that these black holes can be localized on the brane, and they can further be extended into the bulk by a warp function. These solutions have regular horizons and no other curvature singularities appear apart from the string-like ones. The projection of the Gauss-Bonnet term on the brane imposes a constraint relation which requires the presence of matter in the extra dimensions.

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1 Introduction

Recently there has been a growing interest in codimension-2 braneworlds. The most attractive feature of these models is that the vacuum energy (tension) of the brane instead of curving the brane world-volume, merely induces a deficit angle in the bulk solution around the brane [1]. This observation led several people to utilize this property in order to self-tune the effective cosmological constant to zero and provide a solution to the cosmological constant problem [2]. However, soon it was realized [3] that one can only find nonsingular solutions if the brane energy momentum tensor is proportional to its induced metric. To reproduce an effective four-dimensional Einstein equation on the brane one has to introduce a cut-off (brane thickness) [4, 5, 6] with the price of loosing the predictability of the theory. Alternatively, in the thin brane limit four dimensional gravity is recovered as the dynamics of the induced metric on the brane if the gravitational action is modified by the inclusion of either a Gauss-Bonnet term [7] or an induced gravity term on the brane [8].

We are still lacking an understanding of time dependent cosmological solutions in codimension-2 braneworlds. In the thin brane limit, because the energy momentum tensors on the brane and in the bulk are related, the brane equation of state and energy density are tuned and we cannot get the standard cosmology on the brane [9, 10]. One then has to regularize the codimension-2 branes by introducing some thickness and then consider matter on them [11, 12, 13, 14]. To have a cosmological evolution on the regularized branes the brane world-volume should be expanding and in general the bulk space should also evolve in time. This is a formidable task, so an alternatively approach was followed in [15, 16] by considering a codimension-1 brane moving in the regularized static background. The resulting cosmology, however, was unrealistic having a negative Newton's constant (for a review on the cosmology in six dimensions see [17]).

We do not either fully understand black hole solutions on codimension-2 braneworlds. Recently a six-dimensional black hole localized on a 3-brane of codimension-2 [18] was proposed. These solutions are generalization of the $4D$ Aryal, Ford, Vilenkin [19, 20] black hole pierced by a cosmic string adjusted to the codimension-2 branes with a conical structure in the bulk and deformations accommodating the deficit angle. However, it is not clear how to realize these solutions in the thin brane limit where high curvature terms are needed to accommodate matter on the brane. Generalizations to include rotations were presented in [21] and perturbative analysis of these black holes were carried out in [22, 23].

The localization of a black hole on the brane and its extension to the bulk is a difficult task. In codimension-1 braneworlds the first attempt was to consider the Schwarzschild metric and study its black string extension into the bulk [24]. Unfortunately, as suspected by the authors, this string is unstable to classical linear perturbations [25] (for a recent review see [26]). Since then, several authors have attempted to find the full metric using numerical techniques [27]. Analytically, the brane metric equations of motion were considered with the only bulk input coming from the projection of the Weyl tensor [28] onto the brane. Since this system is not closed because it contains an unknown bulk dependent term, assumptions have to be made either in the form of the metric or in the Weyl term [29]. So far there is no clear evidence of what the brane black hole metric is, however, some interesting features which do occur are wormholes and singular horizons [30, 31]. Analysis

of the stability and thermodynamics of these solutions were worked out in [32].

A lower dimensional version of a black hole living on a (2+1)-dimensional braneworld was considered in [33] by Emparan, Horowitz, and Myers. They based their analysis on the so-called C-metric [34] modified by a cosmological constant term. They found a BTZ black hole [38] on the brane which can be extended as a BTZ black string in a four-dimensional AdS bulk. Their thermodynamical stability analysis showed that the black string remains a stable configuration when its transverse size is comparable to the four-dimensional AdS radius, being destabilized by the Gregory-Laflamme instability [25] above that scale, breaking up to a BTZ black hole on a 2-brane.

Three-dimensional gravity, because of its simplicity, is widely recognized as a useful laboratory to study important issues of general relativity. Earlier work on (2+1)-gravity [35, 36] has been followed by many authors studying various aspects of classical and quantum gravity (for a review see [37]). In spite of the fact that general relativity in (2+1) dimensions has neither Newtonian limit nor propagating degrees of freedom, a black hole solution was found (BTZ black hole [38]). The BTZ black hole differs from the Schwarzschild and Kerr solutions in some important aspects: it has a conical-like axially symmetric metric, it is asymptotically anti-de Sitter rather than asymptotically flat, and it has no curvature singularity at the origin. Nonetheless, it is clearly a black hole: it has an event horizon and (in the rotating case) an inner horizon, it appears as the final state of collapsing matter, and it has thermodynamic properties much like those of a (3+1)-dimensional black hole. A singular solution at the origin was presented in [41] as a result of the coupling of BTZ black hole to a conformal matter field, and it was further extended in [42].

In our previous work [39] we studied black holes on an infinitely thin conical 2-brane and their extension into a five-dimensional bulk with a Gauss-Bonnet term. We had found two classes of solutions. The first class consists of the familiar BTZ black hole which solves the junction conditions on a conical 2-brane in vacuum. These solutions in the bulk are BTZ string-like objects with regular horizons and no pathologies. The warping to five-dimensions depends on the length $\sqrt{\alpha}$ where α is the Gauss-Bonnet coupling, and this length scale defines the shape of the horizon. Consistency of the bulk solutions requires a fine-tuned relation between the Gauss-Bonnet coupling and the five-dimensional cosmological constant. The second class of solutions consists of BTZ black holes with short distance corrections. These solutions correspond to a BTZ black hole conformally dressed with a scalar field [41, 42]. Localization of these black holes on the 2-brane leads to the interesting result that the energy-momentum tensor required to support such solutions on the brane corresponds to the energy-momentum tensor of a scalar field in the limit $r/L_3 << 1$, where L_3 is the length scale of the three-dimensional AdS space and r the radial distance on the brane. Also these solutions have black string-like extensions into the bulk.

In this work we generalize our previous work to black objects in six-dimensional braneworlds of codimension-2. We find solutions of four-dimensional Schwarzschild-AdS black holes on the brane which in the six-dimensional spacetime look like black string-like objects with regular horizons. The warping to extra dimensions depends on the Gauss-Bonnet coupling which is fine-tuned to the six-dimensional cosmological constant. In the case of constant deficit angle the localization of the four-dimensional black hole requires matter in the two extra dimensions. The energy-momentum tensor corresponding to this matter

scales as $1/r^6$. This fact defines a length scale in the six-dimensional spacetime above which we recover the standard four-dimensional General Relativity (GR), while at small distances GR is strongly modified. There are also solutions with variable deficit angle, in which case matter is also necessary in the other directions. However, consistency of the bulk equations requires the deficit angle to be constant.

The presence of the Gauss-Bonnet term in codimension-2 braneworlds has important consequences in our solutions. Its projection on the brane gives a consistency relation [8] that dictates the form of the solutions. It allows black string solutions in five dimensions and in six dimensions it specifies the kind of matter which is needed in the bulk in order to support a black hole solution on the brane.

The paper is organized as follows. In section 2 we present in a self-contained way the BTZ string-like solutions of the five-dimensional case. In section 3 we discuss the black string-like solutions of the six-dimensional Einstein equations for constant and variable deficit angles. To complete our solutions we introduce branes, and solving the junction equations we find the conditions to localize the black holes on these branes. In section 4 we discuss the special rôle played by the Gauss-Bonnet term, and finally, in section 5 we conclude.

2 BTZ String-Like Solutions in Five-Dimensional Braneworlds of Codimension-2

We consider the following gravitational action in five dimensions with a Gauss-Bonnet term in the bulk and an induced three-dimensional curvature term on the brane

$$\begin{aligned} S_{\rm grav} &= \frac{M_5^3}{2}\left\{\int d^5x\sqrt{-g^{(5)}}\left[R^{(5)}+\alpha\left(R^{(5)2}-4R^{(5)}_{MN}R^{(5)MN}+R^{(5)}_{MNKL}R^{(5)MNKL}\right)\right]\right. \\ &+ \left. r_c^2\int d^3x\sqrt{-g^{(3)}}\,R^{(3)}\right\}+\int d^5x\mathcal{L}_{bulk}+\int d^3x\mathcal{L}_{brane}\,, \end{aligned} \tag{2.1}$$

where $\alpha\,(\geq 0)$ is the GB coupling constant and $r_c = M_3/M_5^3$ is the induced gravity "cross-over" scale (marking the transition from 3D to 5D gravity). In the above action M_5 is the five-dimensional Planck mass and M_3 is the three-dimensional one. The above induced term has been written in the particular coordinate system in which the metric is

$$ds_5^2 = g_{\mu\nu}(x,\rho)dx^\mu dx^\nu + a^2(x,\rho)d\rho^2 + L^2(x,\rho)d\theta^2\ , \tag{2.2}$$

where $g_{\mu\nu}(x,0)$ is the braneworld metric and x^μ denote three dimensions, $\mu = 0,1,2$ whereas ρ,θ denote the radial and angular coordinates of the two extra dimensions (the ρ direction may or may not be compact and the θ coordinate ranges form 0 to 2π). Capital M, N indices will take values in the five-dimensional space. Note that we have assumed that there exists an azimuthal symmetry in the system, so that both the induced three-dimensional metric and the functions a and L do not depend on θ.

The Einstein equations resulting from the variation of the action (2.1) are

$$G^{(5)N}_M + r_c^2 G^{(3)\nu}_\mu g^\mu_M g^N_\nu \frac{\delta(\rho)}{2\pi L} - \alpha H^N_M = \frac{1}{M_5^3}\left[T^{(B)N}_M + T^{(br)\nu}_\mu g^\mu_M g^N_\nu \frac{\delta(\rho)}{2\pi L}\right]\ , \tag{2.3}$$

where

$$H_M^N = \left[\frac{1}{2}g_M^N(R^{(5)\ 2} - 4R_{KL}^{(5)\ 2} + R_{ABKL}^{(5)\ 2}) - 2R^{(5)}R_M^{(5)N} \right.$$
$$\left. +4R_{MP}^{(5)}R_{(5)}^{NP} \quad +4R_{KMP}^{(5)\quad N}R_{(5)}^{KP} - 2R_{MKLP}^{(5)}R_{(5)}^{NKLP}\right] . \qquad (2.4)$$

To obtain the braneworld equations we expand the metric around the brane as

$$L(x,\rho) = \beta(x)\rho + O(\rho^2) . \qquad (2.5)$$

At the boundary of the internal two-dimensional space where the 2-brane is situated the function L behaves as $L'(x,0) = \beta(x)$, where a prime denotes derivative with respect to ρ. We also demand that the space in the vicinity of the conical singularity is regular which imposes the supplementary conditions that $\partial_\mu\beta = 0$ and $\partial_\rho g_{\mu\nu}(x,0) = 0$ [7].

The extrinsic curvature in the particular gauge $g_{\rho\rho} = 1$ that we are considering is given by $K_{\mu\nu} = g'_{\mu\nu}$. The above decomposition will be helpful in the following for finding the induced dynamics on the brane. We will now use the fact that the second derivatives of the metric functions contain δ-function singularities at the position of the brane. The nature of the singularity then gives the following relations [7]

$$\frac{L''}{L} = -(1-L')\frac{\delta(\rho)}{L} + \mathrm{non-singular\ terms} , \qquad (2.6)$$
$$\frac{K'_{\mu\nu}}{L} = K_{\mu\nu}\frac{\delta(\rho)}{L} + \mathrm{non-singular\ terms} . \qquad (2.7)$$

From the above singularity expressions and using the Gauss-Codazzi equations, we can match the singular parts of the Einstein equations (2.3) and get the following "boundary" Einstein equations

$$G_{\mu\nu}^{(3)} = \frac{1}{M_{(5)}^3(r_c^2 + 8\pi(1-\beta)\alpha)}T_{\mu\nu}^{(br)} + \frac{2\pi(1-\beta)}{r_c^2 + 8\pi(1-\beta)\alpha}g_{\mu\nu} . \qquad (2.8)$$

Note that in the above boundary Einstein equations, as a result of the Gauss-Codazzi reduction procedure, there will also appear terms proportional to the extrinsic curvature and terms coming from the GB term in the bulk. However, if we allow only conical singularities there is no contribution from these terms [7] (see next section for the most general case). Also observe, that the presence of the induced gravity on the brane or the GB term in the bulk is necessary in order to have a non zero energy momentum tensor on the brane.

We assume that there is a localized (2+1) black hole on the brane. The brane metric is

$$ds_3^2 = \left(-n(r)^2dt^2 + n(r)^{-2}dr^2 + r^2d\phi^2\right) , \qquad (2.9)$$

where $0 \le r < \infty$ is the radial coordinate, and ϕ has the usual periodicity $(0, 2\pi)$. We will look for black string solutions of the Einstein equations (2.3) using the five-dimensional metric (2.2) in the form

$$ds_5^2 = f^2(\rho)\left(-n(r)^2dt^2 + n(r)^{-2}dr^2 + r^2d\phi^2\right) + a^2(r,\rho)d\rho^2 + L^2(r,\rho)d\theta^2 . \qquad (2.10)$$

The space outside the conical singularity is regular, therefore, we demand that the warp function $f(\rho)$ is also regular everywhere. We assume that there is only a cosmological constant Λ_5 in the bulk and we take $a(r,\rho)=1$. Then, from the bulk Einstein equations

$$G^{(5)}_{MN} - \alpha H_{MN} = -\frac{\Lambda_5}{M_5^3} g_{MN} \ , \tag{2.11}$$

combining the $(rr, \phi\phi)$ equations we get

$$\left(\dot{n}^2 + n\ddot{n} - \frac{n\dot{n}}{r}\right)\left(1 - 4\alpha \frac{L''}{L}\right) = 0 \ , \tag{2.12}$$

while a combination of the $(\rho\rho, \theta\theta)$ equations gives

$$\left(f'' - \frac{f'L'}{L}\right)\left[3 - 4\frac{\alpha}{f^2}\left(\dot{n}^2 + n\ddot{n} + 2\frac{n\dot{n}}{r} + 3f'^2\right)\right] = 0 \ , \tag{2.13}$$

where a dot denotes derivatives with respect to r. The solutions of the equations (2.12) and (2.13) are summarized in the following table [39]

$n(r)$	$f(\rho)$	$L(\rho)$	$-\Lambda_5$	Constraints
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\forall L(\rho)$	$\frac{3}{4\alpha}$	$L_3^2 = 4\,\alpha$
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	-
BTZ	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	$L_3^2 = 4\,\alpha$
BTZ	± 1	$\frac{1}{\gamma}\,\sinh\left(\gamma\,\rho\right)$	$\frac{3}{l^2}$	$\gamma = \sqrt{-\frac{2\Lambda_5}{3+4\alpha\Lambda_5}}$
$\forall n(r)$	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	-
$\sqrt{-M + \frac{r^2}{L_3^2} - \frac{\zeta}{r}}$	$\cosh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{3}{4\alpha}$	$L_3^2 = 4\,\alpha$
$\sqrt{-M + \frac{r^2}{L_3^2} - \frac{\zeta}{r}}$	± 1	$2\,\beta\,\sqrt{\alpha}\,\sinh\left(\frac{\rho}{2\sqrt{\alpha}}\right)$	$\frac{1}{4\alpha}$	$\Lambda_5 = -\frac{1}{4\alpha} = -\frac{3}{L_3^2}$

Table 1: BTZ String-Like Solutions in Five-Dimensional Braneworlds of Codimension-2

In the above table L_3 is the length scale of AdS_3 space. Note that in all solutions there is a fine-tuned relation between the Gauss-Bonnet coupling α and the five-dimensional cosmological constant Λ_5, except for the solution in the fourth row. Also observe that the solution in the third row is a kind of combination of the solutions in the first and second row. This is a result of the way we solve the factorized equations (2.12) and (2.13) [39].

To introduce a brane we must solve the corresponding junction conditions given by the Einstein equations on the brane (2.8) using the induced metric on the brane given by (2.9). For the case when $n(r)$ corresponds to the BTZ black hole $n^2(r) = -M + \frac{r^2}{L_3^2}$, and the brane cosmological constant is given by $\Lambda_3 = -1/L_3^2$, we found that the energy-momentum tensor is null. Therefore, the BTZ black hole is localized on the brane in vacuum.

When $n(r)$ is of the form given by

$$n(r) = \sqrt{-M + \frac{r^2}{L_3^2} - \frac{\zeta}{r}} \ , \tag{2.14}$$

which is the BTZ black hole solution with a short distance correction term, we can go back to (2.8) and solve for $T^{br}_{\mu\nu}$. Then we find that the matter source necessary to sustain such a solution on the brane is given by

$$T^{\beta}_{\alpha} = \text{diag}\ \left(\frac{\zeta}{2r^3}, \frac{\zeta}{2r^3}, -\frac{\zeta}{r^3}\right) \ , \tag{2.15}$$

which is conserved on the brane [40]. Interesting enough, for a scalar field conformally coupled to BTZ [41, 42], the energy-momentum tensor needed to support such a solution at a certain limit reduces to (2.15) which is necessary to localize this black hole on the conical 2-brane.

These solutions extend the brane BTZ black hole into the bulk. Calculating the square of the Riemann tensor we find that at the AdS horizon ($\rho \to \infty$) all solutions give finite result and hence the only singularity is the BTZ-corrected black hole singularity extended into the bulk. The warp function $f^2(\rho)$ gives the shape of a 'throat' to the horizon of the BTZ string-like solution. The size of the horizon is defined by the scale $\sqrt{\alpha}$ and this scale is fine-tuned to the length scale of the five-dimensional AdS space.

3 Black String-Like solutions in Six-Dimensional Braneworlds of Codimension-2

In this section we will look for black string solutions in six-dimensions with conical singularities. We consider the gravitational action (2.1) in six dimensions

$$\begin{aligned} S_{grav} &= \frac{M_6^4}{2}\left\{\int d^6x\sqrt{-g^{(6)}}\left[R^{(6)} + \alpha\left(R^{(6)2} - 4R^{(6)}_{MN}R^{(6)MN} + R^{(6)}_{MNKL}R^{(6)MNKL}\right)\right]\right. \\ &+ \left. r_c^2\int d^4x\sqrt{-g^{(4)}}R^{(4)}\right\} + \int d^6x\mathcal{L}_{bulk} + \int d^4x\mathcal{L}_{brane} \,. \end{aligned} \tag{3.1}$$

Here $r_c = M_4/M_6^2$ is the induced gravity "cross-over" scale (marking the transition from 4D to 6D gravity), M_6 is the six-dimensional Planck mass and M_4 is the four-dimensional one.

The metric as in the five-dimensional case is

$$ds_6^2 = g_{\mu\nu}(r,\chi)dx^\mu dx^\nu + a^2(r,\chi)d\chi^2 + L^2(r,\chi)d\xi^2 \ , \tag{3.2}$$

now with $\mu = 0,1,2,3$ whereas χ, ξ denote the radial and angular coordinates of the two extra dimensions (the χ direction may or may not be compact and the ξ coordinate ranges form 0 to 2π). Note that we have assumed that there exists an azimuthal symmetry in the system, so that both the induced four-dimensional metric and the functions a and L do not depend on ξ.

The corresponding Einstein equations are

$$G_M^{(6)N} + r_c^2 G_\mu^{(4)\nu} g_M^\mu g_\nu^N \frac{\delta(\chi)}{2\pi L} - \alpha H_M^N = \frac{1}{M_6^4}\left[-\Lambda_6 + T_M^{(B)N} + T_\mu^{(br)\nu} g_M^\mu g_\nu^N \frac{\delta(\chi)}{2\pi L}\right] , \tag{3.3}$$

where H_M^N is the corresponding six-dimensional term of (2.4) To obtain the braneworld equations we expand the metric around the 3-brane as

$$L(r,\chi) = \beta(r)\chi + O(\chi^2) , \tag{3.4}$$

and as in the five-dimensional case the function L behaves as $L'(r,0) = \beta(r)$, where a prime now denotes derivative with respect to χ. The "boundary" Einstein equations are

$$\begin{aligned} G_{\mu\nu}^{(4)}\left(r_c^2 + 8\pi(1-\beta)\alpha\right)|_0 &= \frac{1}{M_6^4} T_{\mu\nu}^{(br)}|_0 + 2\pi(1-\beta) g_{\mu\nu}|_0 \\ &+ \pi L(r,\chi)\, E_{\mu\nu}|_0 - 2\pi\beta\alpha\, W_{\mu\nu}|_0 , \end{aligned} \tag{3.5}$$

where the term

$$E_{\mu\nu}|_0 = \left(K_{\mu\nu} - g_{\mu\nu} K\right)|_0 , \tag{3.6}$$

appears because of the presence of the induced gravity term in the gravitational action, while the term

$$\begin{aligned} W_{\mu\nu}|_0 &= g^{\lambda\sigma}\partial_\chi g_{\mu\lambda}\partial_\chi g_{\nu\sigma}|_0 - g^{\lambda\sigma}\partial_\chi g_{\lambda\sigma}\partial_\chi g_{\mu\nu}|_0 \\ &+ \frac{1}{2} g_{\mu\nu}\left[\left(g^{\lambda\sigma}\partial_\chi g_{\lambda\sigma}\right)^2 - g^{\lambda\sigma} g^{\delta\rho}\partial_\chi g_{\lambda\delta}\partial_\chi g_{\sigma\rho}\right]\Big|_0 , \end{aligned} \tag{3.7}$$

is the Weyl term due to the presence of the Gauss-Bonnet term in the bulk [7]. The effective four-dimensional mass and cosmological constant are

$$M_{Pl}^2 = M_6^4 (r_c^2 + 8\pi(1-\beta)\alpha) , \tag{3.8}$$
$$\Lambda_4 = \lambda - 2\pi M_6^4 (1-\beta) , \tag{3.9}$$

where λ is the brane tension.

If we demand that the space in the vicinity of the conical singularity is regular ($\partial_\mu\beta = 0$) then (3.5) simply becomes [7, 8]

$$G_{\mu\nu}^{(4)}\left(r_c^2 + 8\pi(1-\beta)\alpha\right)|_0 = \frac{1}{M_6^4} T_{\mu\nu}^{(br)}|_0 + 2\pi(1-\beta) g_{\mu\nu}|_0 . \tag{3.10}$$

3.1 Black String-Like Solutions with pure conical singularity: Case $\partial_\mu\beta = 0$

In this subsection we make the assumption that the singularity is purely conical. Thus, we will solve the bulk equations with a constant deficit angle β. We assume that the brane metric is

$$ds_4^2 = -A(r)^2 dt^2 + A(r)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 , \tag{3.11}$$

where $0 \leq r < \infty$ is the radial coordinate, and ϕ has the usual periodicity $(0, 2\pi)$. We will look for black string solutions of the Einstein equations (3.3) using the six-dimensional metric (3.2) in the form

$$ds_6^2 = F^2(\chi)\left(-A(r)^2 dt^2 + A(r)^{-2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right) + a^2(r,\chi) d\chi^2 + L^2(r,\chi) d\xi^2 \ . \tag{3.12}$$

As we discussed in the previous section, the space outside the conical singularity is regular, therefore we demand that the warp function $F(\chi)$ is also regular everywhere. We have split the general bulk energy momentum tensor $\tilde{T}_M^{(B)\,N}$ into a cosmological constant Λ_6 and a bulk energy momentum tensor $T_M^{(B)\,N}$. Moreover, we take $a(r,\chi) = 1$. Then from the bulk equations

$$G_{MN}^{(6)} - \alpha H_{MN} = \frac{1}{M_6^4}\left(-\Lambda_6 \, g_{MN} + T_{MN}^{(B)}\right) \ , \tag{3.13}$$

by taking the combination $rr - \theta\theta$ and $\chi\chi - \xi\xi$ of the Einstein equations we respectively get

$$\left(\dot{A}^2 + A\ddot{A} - \frac{A^2}{r^2} + \frac{1}{r^2}\right)\left[1 - 4\alpha\left(\frac{L''}{L} + \frac{F''}{F} + \frac{F'L'}{FL}\right)\right] = 0 \ , \tag{3.14}$$

$$\left(F'' - \frac{F'L'}{L}\right)\left[1 - \frac{2\alpha}{F^2}\left(\dot{A}^2 + A\ddot{A} + \frac{A^2}{r^2} + 4\frac{A\dot{A}}{r} - \frac{1}{r^2} + 6F'^2\right)\right] = 0 \ . \tag{3.15}$$

The $\chi\chi$ and $\xi\xi$ components of the Einstein equations are

$$\begin{aligned} G_\chi^\chi &= T_\chi^\chi \ , \\ G_\xi^\xi &= T_\xi^\xi \ , \end{aligned} \tag{3.16}$$

and therefore to get (3.15) we must take the difference $G_\chi^\chi - G_\xi^\xi = T_\chi^\chi - T_\xi^\xi$, and as the only remaining energy contribution in the bulk is a cosmological constant, the matter in the extra two dimensions must satisfy the relation $T_\chi^\chi = T_\xi^\xi$.

3.1.1 Black String-Like Solutions of the Bulk Equations: Case 1

We will consider first

$$\dot{A}^2 + A\ddot{A} - \frac{A^2}{r^2} + \frac{1}{r^2} = 0 \ , \tag{3.17}$$

which has as a solution

$$A^2(r) = 1 + \frac{r^2}{L_4^2} - \frac{\zeta}{r} \ , \tag{3.18}$$

where L_4 is the length scale of the AdS_4 space, and ζ is an integration constant. Then equation (3.15) becomes

$$\left(F'' - \frac{F'L'}{L}\right)\left[1 - 12\frac{\alpha}{F^2}\left(\frac{1}{L_4^2} + F'^2\right)\right] = 0 \ . \tag{3.19}$$

From the above equation we have two cases:

- Case 1a: The first case is

$$F'^2 - \frac{F^2}{12\alpha} + \frac{1}{L_4^2} = 0 \ . \tag{3.20}$$

This equation has the following solution

$$F(\chi) = C_1\, e^{\frac{\chi}{2\sqrt{3\alpha}}} + C_2\, e^{\frac{-\chi}{2\sqrt{3\alpha}}} \ , \tag{3.21}$$

where C_1 and C_2 are integration constants which satisfy the relation $C_1\, C_2 = 3\alpha/L_4^2$. The function $F(\chi)$ is regular and if we require on the position of the brane the boundary condition $F^2(\chi = 0) = 1$, the integration constants can be expressed in terms of α and L_4 as

$$\begin{aligned} C_1 &= \pm\frac{1 + \varepsilon\sqrt{1 - 12\frac{\alpha}{L_4^2}}}{2} \ , \\ C_2 &= \pm\frac{1 - \varepsilon\sqrt{1 - 12\frac{\alpha}{L_4^2}}}{2} \ , \end{aligned} \tag{3.22}$$

where $\varepsilon = \pm 1$ independently of the $\pm$ sign in C_1 and C_2. Moreover, we need $\partial_\chi g_{\mu\nu}|_{(\chi=0)} = 0$, i.e., $\partial_\chi F|_{(\chi=0)} = 0$, therefore $C_1 = C_2 = \frac{1}{2}$ and $\alpha = \frac{L_4^2}{12}$. Thus, $F(\chi) = \cosh\left(\frac{\chi}{L_4}\right)$. Substituting the above solutions into the tt, rr, $\theta\theta$, and $\phi\phi$ components of the Einstein equations we get a fine-tuned relation between α and Λ_6

$$\Lambda_6 = -\frac{5}{12\alpha} = -\frac{5}{L_4^2} \ . \tag{3.23}$$

Because of the positivity of α the six-dimensional bulk space is Anti-de-Sitter. In addition we have a relation between the six-dimensional cosmological constant Λ_6 and the AdS_4 length scale L_4. If we require the bulk equations to have as a solution the Schwarzschild-AdS black hole (3.18) with $\zeta \neq 0$ then consistency of the bulk equations requires the $\chi\chi$ and $\xi\xi$ components of the energy-momentum tensor to have the form

$$T_\chi^{(B)\,\chi} = T_\xi^{(B)\,\xi} = -\frac{6\,\alpha\,\zeta^2}{r^6}\,\frac{1}{F(\chi)^4} \ , \tag{3.24}$$

with the other components to be zero. Notice that except for the warp function $F(\chi)$ these components of the energy-momentum tensor do not depend explicitly on χ but only on the radial coordinate on the brane. We will give a detailed account of this dependence in the next section.

- Case 1b: The second case is to consider

$$F'' - \frac{F'L'}{L} = 0 \ , \tag{3.25}$$

which means that $L(\chi) = L_0\, F'(\chi)$. In any subcase we recover in the same way the results of case 1a. However $L(\chi)$ is no more arbitrary and is given by $L(\chi) = \beta\, L_4 \sinh\left(\frac{\chi}{L_4}\right)$, where we used the boundary conditions $L(\chi = 0) = 0$ and $L'(\chi = 0) = \beta$.

- There are also two constant solutions for $F(\chi)$ which read

$$
\begin{aligned}
F(\chi) &= \pm 1, && (3.26)\\
A(r) &= \sqrt{1+\frac{r^2}{L_4^2}-\frac{\zeta}{r}}, && (3.27)\\
L(\chi) &= \beta\,\frac{\sinh(\gamma\,\chi)}{\gamma}, && (3.28)\\
\text{with}\quad \gamma &= \frac{1}{L_4}\sqrt{\frac{1-\frac{L_4^2}{4\alpha}}{1-\frac{L_4^2}{12\alpha}}} &&\\
\Lambda_6 &= -\frac{6}{L_4^2}\left(1-\frac{2\alpha}{L_4^2}\right) && (3.29)\\
\text{for}\quad \zeta\neq 0\quad T_\chi^{(B)\,\chi} &= T_\xi^{(B)\,\xi} = -\frac{6\alpha\zeta^2}{r^6}. && (3.30)
\end{aligned}
$$

The second one has the same $F(\chi)$ and $A(r)$ functions, as well as $T_\chi^{(B)\,\chi} = T_\xi^{(B)\,\xi}$, but

$$
\begin{aligned}
L(\chi) &= \beta\,\chi\,\frac{\sinh\gamma}{\gamma}, && (3.31)\\
T_t^{(B)\,t} = T_r^{(B)\,r} = T_\theta^{(B)\,\theta} &= T_\phi^{(B)\,\phi} = \frac{3\,(4\alpha-L_4^2)}{L_4^4}. && (3.32)
\end{aligned}
$$

3.1.2 Black String-Like Solutions of the Bulk Equations: Case 2

In this section we chose from (3.14) the equation

$$1-4\alpha\left(\frac{L''}{L}+\frac{F''}{F}+\frac{F'L'}{FL}\right)=0\ . \tag{3.33}$$

- Case 2a: The first one is to consider from (3.15)

$$F''-\frac{F'L'}{L}=0\ , \tag{3.34}$$

which means that $L(\chi) = L_0\,F'(\chi)$. Although this is not enough to solve (3.33), there is an exponential solution for both functions given by

$$
\begin{aligned}
F(\chi) &= C_1\,e^{\frac{\chi}{2\sqrt{3\alpha}}} + C_2\,e^{\frac{-\chi}{2\sqrt{3\alpha}}}\ , && (3.35)\\
L(\chi) &= L_0\,F'(\chi)\,. && (3.36)
\end{aligned}
$$

Substituting the above solution into the tt, rr, $\theta\theta$, and $\phi\phi$ components of the Einstein equations we get a fine-tuned relation between α and Λ_6

$$\Lambda_6 = -\frac{5}{12\alpha}\ . \tag{3.37}$$

If we choose as in the first case $C_1C_2 = \frac{3\alpha}{L_4^2}$ then we get a differential equation for $A(r)$, which has the following solution

$$A^2(r) = 1 + \frac{r^2}{L_4^2} \pm \sqrt{1 + \frac{C_3}{L_4^2} + \frac{C_4}{L_4^4}\, r}\ , \tag{3.38}$$

with the constraint $12\alpha = L_4^2$ which relates the 6-dimensional cosmological constant Λ_6 and the AdS_4 length scale L_4 as $\Lambda_6 = -\frac{5}{L_4^2}$, therefore imposing the boundary conditions $F^2(\chi = 0) = 1$ ($\partial_\chi F|_{(\chi=0)} = 0$ is already satisfied), $L(\chi = 0) = 0$ and $L'(\chi = 0) = \beta$ we have $F(\chi) = \cosh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$ and $L(\chi) = 2\sqrt{3\alpha}\,\beta\,\sinh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$ which satisfy all Einstein equations.

- Case 2b: The second case is to consider from (3.15)

$$\left(F^2 - 12\alpha\, F'^2\right) - 2\alpha\left(\dot{A}^2 + A\ddot{A} + \frac{A^2}{r^2} + 4\frac{A\dot{A}}{r} - \frac{1}{r^2}\right) = 0\ , \tag{3.39}$$

The first term is a function of χ while the second one is a function of r. Therefore, each term should be, in general, equal to a constant κ. We then have

$$F^2 - 12\alpha\, F'^2 = \kappa\ , \tag{3.40}$$

$$2\alpha\left(\dot{A}^2 + A\ddot{A} + \frac{A^2}{r^2} + 4\frac{A\dot{A}}{r} - \frac{1}{r^2}\right) = \kappa\ , \tag{3.41}$$

which give

$$F(\chi) = C_1\, e^{\frac{\chi}{2\sqrt{3\alpha}}} + C_2\, e^{\frac{-\chi}{2\sqrt{3\alpha}}}\ , \tag{3.42}$$

$$A^2(r) = 1 + \frac{2\,C_3}{r^2} + \frac{C_4}{r} + \frac{\kappa\, r^2}{12\,\alpha}, \tag{3.43}$$

with $C_1C_2 = \frac{\kappa}{4}$. No solution can be found unless we set $\Lambda_6 = -\frac{5}{12\alpha}$. Then we need to solve the following differential equation

$$-L(\chi) + \sqrt{3\alpha}\left(\frac{1 - \frac{\kappa}{4\,C_1^2}\, e^{\frac{-\chi}{\sqrt{3\alpha}}}}{1 + \frac{\kappa}{4\,C_1^2}\, e^{\frac{-\chi}{\sqrt{3\alpha}}}}\right) L'(\chi) + 6\alpha\, L''(\chi) = 0, \tag{3.44}$$

which has the following solution

$$L(\chi) = \frac{4\,C_1^2\, C_5\, e^{\frac{\chi}{\sqrt{3\alpha}}}}{\kappa}\ {}_2F_1\left[\frac{1}{2}, 2, \frac{5}{2}, -\frac{4\,C_1^2}{\kappa}\, e^{\frac{\chi}{\sqrt{3\alpha}}}\right]\ , \tag{3.45}$$

being ${}_2F_1$ the hypergeometric function of the second kind. The $\chi\chi$ and $\xi\xi$ components of the Einstein equations impose us $C_1 = 0$, but then we cannot have $\partial_\chi\, g_{\mu\nu} = 0$.

Therefore we must have $C_1 \neq 0$ and we have to consider a specific r and χ dependent $\chi\chi$ and $\xi\xi$ components of the energy momentum tensor given by

$$T_\chi^{(B)\,\chi} = T_\xi^{(B)\,\xi} = -\frac{2\alpha\,(40\,C_3^2 + 24\,C_3\,C_4\,r + 3\,C_4^2\,r^2)}{r^8}\,\frac{1}{F(\chi)^4}. \tag{3.46}$$

Imposing the boundary conditions $F^2(\chi=0)=1$, $L(\chi=0)=0$ and $L'(\chi=0)=\beta$ we get

$$\begin{aligned}
C_1 &= \pm\frac{1+\varepsilon\sqrt{1-\kappa}}{2}\,, && (3.47)\\
C_2 &= \pm\frac{1-\varepsilon\sqrt{1-\kappa}}{2}\,, && (3.48)\\
\kappa &= \beta, && (3.49)\\
C_5 &= \frac{5\,\sqrt{3\alpha}\,\beta^3}{\eta^2}\Big(5\,\beta\;{}_2F_1\left[\frac{1}{2},2,\frac{5}{2},-\frac{\eta^2}{\beta}\right]\\
&- 2\,\eta^2\;{}_2F_1\left[\frac{3}{2},3,\frac{7}{2},-\frac{\eta^2}{\beta}\right]\Big), && (3.50)\\
\text{with}\quad \eta &= 1+\sqrt{1-\beta}.
\end{aligned}$$

Moreover, we must have $\partial_\chi F|_{(\chi=0)} = 0$ therefore $C_1 = C_2 = \frac{1}{2}$ i.e. $\kappa = 1$ which imposes $C_5 = 0$. Therefore there is no solution.

- There is also a constant solution for $F(\chi)$ which gives

$$\begin{aligned}
F(\chi) &= \pm 1, && (3.51)\\
A(r)^2 &= 1+\frac{r^2}{4\alpha}-\frac{\sqrt{3}}{12\alpha}\sqrt{2\,r^4 - 3\,C_4\,r + 48\,\alpha\,(\alpha - C_3)}, && (3.52)\\
L(\chi) &= 2\sqrt{\alpha}\,\beta\,\sinh\frac{\chi}{2\sqrt{\alpha}}, && (3.53)\\
\Lambda_6 &= -\frac{1}{4\alpha}. && (3.54)
\end{aligned}$$

3.1.3 Localization of the Bulk Black Hole on the Brane

In order to complete our solution with the introduction of the brane we must solve the corresponding junction conditions given by the Einstein equations on the brane (3.10) using the induced metric on the brane given by (3.11).

Equation (3.10) can be written as

$$\frac{T_\mu^{(br)\,\nu}|_0}{M_4^6} = \left(r_c^2 + 8\pi\,(1-\beta)\,\alpha\right)\,G_\mu^{(4)\,\nu}|_0 - 2\pi\,(1-\beta)\,g_\mu{}^\nu|_0, \tag{3.55}$$

Moreover, the $(\chi\chi)$ component of the six-dimensional Einstein tensor evaluated at $\chi=0$ is

$$-\frac{1}{2}\,R^{(4)}|_0 - \frac{\alpha}{2}\left(R^{(4)\,2} - 4R_{\mu\nu}^{(4)\,2} + R_{\mu\nu\kappa\lambda}^{(4)\,2}\right)\Big|_0 = \frac{1}{M_6^4}\,T_\chi^{(B)\,\chi}|_0 - \frac{\Lambda_6}{M_6^4}|_0, \tag{3.56}$$

which gives the form of the $(\chi\chi)$ component of the bulk energy momentum tensor in terms of brane quantities

$$\frac{1}{M_6^4}\, T_\chi^{(B)\,\chi}|_0 = -\frac{1}{2}\, R^{(4)}|_0 - \frac{\alpha}{2}\left(R^{(4)\,2} - 4R_{\mu\nu}^{(4)\,2} + R_{\mu\nu\kappa\lambda}^{(4)\,2}\right)\Big|_0 + \frac{\Lambda_6}{M_6^4}|_0. \tag{3.57}$$

- <u>For case 1a we have:</u>

$$\begin{aligned} A^2(r) &= 1 + \frac{r^2}{L_4^2} - \frac{\zeta}{r}, \\ F(\chi) &= \cosh\left(\frac{\chi}{2\sqrt{3\alpha}}\right), \end{aligned}$$

In this case $L(\chi)$ is arbitrary, and we have the constraint $\alpha = \frac{L_4^2}{12}$. In addition,

$$\Lambda_6 = -\frac{5}{12\alpha} = -\frac{5}{L_4^2}, \tag{3.58}$$

$$T_\chi^{(B)\,\chi} = T_\xi^{(B)\,\xi} = -\frac{6\,\alpha\,\zeta^2}{r^6}\,\frac{1}{F(\chi)^4}. \tag{3.59}$$

Then (3.57) is consistent with (3.59) whereas (3.55) gives

$$\frac{T_\mu^\nu}{M_6^4} = 3\frac{r_c^2}{L_4^2}\,\delta_\mu^\nu. \tag{3.60}$$

- <u>For case 1b we have:</u>

$$A^2(r) = 1 + \frac{r^2}{L_4^2} - \frac{\zeta}{r},$$

$$F(\chi) = \cosh\left(\frac{\chi}{L_4}\right), \tag{3.61}$$

$$L(\chi) = \beta\, L_4\, \sinh\left(\frac{\chi}{L_4}\right), \tag{3.62}$$

with the constraint $\alpha = \frac{L_4^2}{12}$ and

$$\begin{aligned} \Lambda_6 &= -\frac{5}{12\alpha} = -\frac{5}{L_4^2}, \\ T_\chi^{(B)\,\chi} = T_\xi^{(B)\,\xi} &= -\frac{6\,\alpha\,\zeta^2}{r^6}\,\frac{1}{F(\chi)^4}. \end{aligned} \tag{3.63}$$

From (3.55) we get $\frac{T_\mu^\nu}{M_6^4} = 3\frac{r_c^2}{L_4^2}\,\delta_\mu^\nu$ whereas (3.57) and (3.63) are consistent.

- For case 2a we have:

$$\begin{aligned} A^2(r) &= 1+\frac{r^2}{L_4^2} \pm \sqrt{1+\frac{C_3}{L_4^2}+\frac{C_4}{L_4^4}\, r}, \\ F(\chi) &= \cosh\left(\frac{\chi}{2\sqrt{3\alpha}}\right), \\ L(\chi) &= 2\sqrt{3\alpha}\,\beta\,\sinh\left(\frac{\chi}{2\sqrt{3\alpha}}\right), \\ \Lambda_6 &= -\frac{5}{12\alpha} = -\frac{5}{L_4^2}, \end{aligned}$$

In this case (3.57) and (3.55) give respectively some complicated r dependent expressions for T^χ_χ and T^μ_μ, as well as for the solution (3.51)-(3.54).

- For case 2b we have no solution.

Our results are summarized in Table 2.

$A^2(r)$	$F(\chi)$	$L(\chi)$	$-\Lambda_6$	Constraints & $T^{(B)}$
$1+\frac{r^2}{L_4^2}-\frac{\zeta}{r}$	$\cosh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$	$\forall L(\chi)$	$\frac{5}{12\alpha}$	$\alpha=\frac{L_4^2}{12}$, $T^\chi_\chi=T^\xi_\xi=-\frac{6\alpha\zeta^2}{r^6F(\chi)^4}$
$1+\frac{r^2}{L_4^2}-\frac{\zeta}{r}$	$\cosh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$	$2\sqrt{3\alpha}\beta\sinh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$	$\frac{5}{12\alpha}$	$\alpha=\frac{L_4^2}{12}$, $T^\chi_\chi=T^\xi_\xi=-\frac{6\alpha\zeta^2}{r^6F(\chi)^4}$
$1+\frac{r^2}{L_4^2}-\frac{\zeta}{r}$	±1	$\frac{\beta}{\gamma}\sinh(\gamma\,\chi)$	$\frac{6}{L_4^2}\left(1-\frac{2\alpha}{L_4^2}\right)$	$\gamma=\frac{1}{L_4}\sqrt{\frac{1-\frac{L_4^2}{4\alpha}}{1-\frac{L_4^2}{12\alpha}}}$, $T^\chi_\chi=T^\xi_\xi=-\frac{6\alpha\zeta^2}{r^6}$
$1+\frac{r^2}{L_4^2}-\frac{\zeta}{r}$	±1	$\frac{\beta}{\gamma}\chi\sinh\gamma$	$\frac{6}{L_4^2}\left(1-\frac{2\alpha}{L_4^2}\right)$	$\gamma=\frac{1}{L_4}\sqrt{\frac{1-\frac{L_4^2}{4\alpha}}{1-\frac{L_4^2}{12\alpha}}}$, $T^\chi_\chi=T^\xi_\xi=-\frac{6\alpha\zeta^2}{r^6}$, $T^t_t=T^r_r=T^\theta_\theta=T^\phi_\phi=\frac{3(4\alpha-L_4^2)}{L_4^2}$
(3.38)	$\cosh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$	$2\sqrt{3\alpha}\,\beta\,\sinh\left(\frac{\chi}{2\sqrt{3\alpha}}\right)$	$\frac{5}{12\alpha}$	$\alpha=\frac{L_4^2}{12}$
(3.52)	±1	$2\sqrt{\alpha}\,\beta\,\sinh\left(\frac{\chi}{2\sqrt{\alpha}}\right)$	$\frac{1}{4\alpha}$	$\alpha=\frac{L_4^2}{4}$

Table 2: Black String-Like Solutions in Six-Dimensional Braneworlds of Codimension-2

3.2 Curvature singularity: Case $\partial_\mu\beta\neq0$

In this section we relax the assumption of the purely conical singularity. Therefore, in general $\partial_\chi\, g_{\mu\nu}\neq0$ and $\beta(r)$ is a function of r.

3.2.1 Black String-Like Solutions of the Bulk Equations

In this case the combination of the $rr - tt$ components of the bulk equations (3.13) give

$$rr - tt\colon \; -\frac{A^2\,\ddot{L}}{r^2\,F^4\,L}\left[4\alpha - 4\alpha A^2 + r^2\left(F^2 - 4\alpha F'^2 - 8\alpha F F''\right)\right]. \tag{3.64}$$

If we want to keep this factorized form, we can choose $\ddot{L} = 0$ which will not simplify our task. Therefore, we consider in general $\ddot{L} \neq 0$. The other possibility is to consider the term in square brackets equal to zero.

- Case 1: In this case the term in square brackets of (3.64) is equal to zero. Thus we will have $T_r^{(B)\,r} = T_t^{(B)\,t}$ and we need to solve the following equations

$$\begin{aligned} 4l^2\alpha - 4\alpha A^2 &= -\kappa r^2, & (3.65)\\ F^2 - 4\alpha F'^2 - 8\alpha F F'' &= \kappa\,, & (3.66)\end{aligned}$$

  where κ is a constant. Then the solutions are

$$\begin{aligned} A^2(r) &= 1 + \frac{\kappa}{4\alpha}\,r^2, & (3.67)\\ F(\chi) &= C_1\,e^{\frac{\chi}{2\sqrt{3\alpha}}} + C_2\,e^{\frac{-\chi}{2\sqrt{3\alpha}}}, & (3.68)\end{aligned}$$

  with $\kappa = \frac{4\,C_1\,C_2}{3}$. If we redefine $\frac{4\alpha}{\kappa} = L_4^2$ we get

$$\begin{aligned} A^2(r) &= 1 + \frac{r^2}{L_4^2}\,, & (3.69)\\ F(\chi) &= C_1\,e^{\frac{\chi}{2\sqrt{3\alpha}}} + C_2\,e^{\frac{-\chi}{2\sqrt{3\alpha}}} \quad \text{with} \quad C_1\,C_2 = \frac{3\alpha}{L_4^2}\,. & (3.70)\end{aligned}$$

  Then all the bulk Einstein equations are satisfied for $\Lambda_{(6)} = -\frac{5}{12\alpha}$ and with no matter in the bulk. Furthermore, if we require on the position of the brane the boundary condition $F^2(\chi = 0) = 1$, the integration constants can be expressed as in the case $1a$ with constant deficit angle, in terms of α and L_4

$$\begin{aligned} C_1 &= \pm\frac{1 + \varepsilon\sqrt{1 - 12\frac{\alpha}{L_4^2}}}{2}\,, \\ C_2 &= \pm\frac{1 - \varepsilon\sqrt{1 - 12\frac{\alpha}{L_4^2}}}{2}\,, & (3.71)\end{aligned}$$

  where $\varepsilon = \pm 1$ independently of the $\pm$ sign in C_1 and C_2. Moreover, $L(r, \chi)$ is arbitrary.

- Case 2: In this case the factorized equation (3.64) is not equal to zero (i.e. $T_r^{(B)\,r} \neq T_t^{(B)\,t}$) but then the bulk Einstein equations cannot be solved in general. However, if

we consider that $A(r)$ has the same form as in the previous subsection for cases $1a$ and $1b$,

$$A^2(r) = 1 + \frac{r^2}{L_4^2} - \frac{\zeta}{r} \ , \tag{3.72}$$

then the combination $\theta\theta - tt$ and $\chi\chi - \xi\xi$ of the bulk equations (3.13) can respectively be factorized as

$$\begin{aligned}
\theta\theta - tt \quad &: \quad \frac{(2l^2 r - 3\zeta)\,\dot{L}\,[8\alpha\zeta L_4^2 + 4\alpha r^3 + r^3 L_4^2\,(4\alpha F'^2 + 8\alpha F F'' - F^2)]}{2r^5 L_4^2 F^4 L} \\
&\quad - \frac{12\alpha\zeta r \ddot{L}\,(r^3 + l^2 L_4^2 r - L_4^2 \zeta)}{2r^5 L_4^2 F^4 L} \ , \qquad (3.73) \\
\chi\chi - \xi\xi \quad &: \quad \left(12\alpha - L_4^2 F^2 + 12\alpha L_4^2 F'^2\right) \times \\
\times &\frac{\left[\dot{L}\,(4r^3 + 2l^2 L_4^2 r - \zeta L_4^2) + r\,\ddot{L}\,(r^3 + L_4^2\, r\, l^2 - \zeta\, L_4^2) + 4r^2 L_4^2 F\,(F' L' - F'' L)\right]}{r^2 L_4^4 F^4 L} \ , \qquad (3.74)
\end{aligned}$$

where a dot denotes derivatives with respect to r. We note here that in the $\theta\theta - tt$ combination the first term in brackets can never be zero while the second one cannot be solved analytically. Therefore here $T_r^r \neq T_\theta^\theta$. In the $\chi\chi - \xi\xi$ combination the second term in brackets can not also be solved analytically. Therefore the only term which can be equal to zero in order to keep a factorized form is the first bracket in (3.74). Then we get that $F(\chi) = C_1\, e^{\frac{\chi}{2\sqrt{3\alpha}}} + C_2\, e^{\frac{-\chi}{2\sqrt{3\alpha}}}$ with $C_1\, C_2 = \frac{3\alpha}{L_4^2}$. Finally, the bulk Einstein equations are satisfied for $\Lambda_{(6)} = -\frac{5}{12\alpha}$ and for $\zeta = 0$, unless we have the following bulk energy momentum tensor

$$\begin{aligned}
T_t^{(B)\,t} \quad &= \quad -T_\theta^{(B)\,\theta} = -T_\phi^{(B)\,\phi} = \frac{2\alpha\zeta}{r^5 L F^4} \\
&\qquad \times \left[\left(3\zeta - 2l^2 r\right)\dot{L} + 2r^2 \ddot{L}\left(l^2 + \frac{r^2}{L_4^2} - \frac{\zeta}{r}\right)\right] , \qquad (3.75) \\
T_r^{(B)\,r} \quad &= \quad \frac{2\alpha\zeta}{r^5 L F^4}\left(3\zeta - 2l^2 r\right)\dot{L} \ , \qquad (3.76) \\
T_\chi^{(B)\,\chi} \quad &= \quad T_\xi^{(B)\,\xi} = -\frac{6\,\alpha\,\zeta^2}{r^6}\,\frac{1}{F(\chi)^4} \ . \qquad (3.77)
\end{aligned}$$

Furthermore, if we require on the position of the brane the boundary condition $F^2(\chi = 0) = 1$, the integration constants can be expressed as in (3.71). Moreover, $L(r, \chi)$ is arbitrary.

3.2.2 Localization of the Bulk Black Hole on the Brane

In order to complete our solution with the introduction of the brane we must solve the corresponding junction conditions given by the Einstein equations on the brane (3.5) using

the induced metric on the brane given by (3.11). Equation (3.5) can be written as

$$\frac{T^{(br)\,\nu}_{\mu}|_0}{M_4^6} = \left(r_c^2 + 8\pi(1-\beta)\alpha\right) G^{(4)\,\mu}_{\nu}|_0 - 2\pi(1-\beta)\delta^{\mu}_{\nu}|_0 - \pi L(r,\chi)\, E^{\mu}_{\nu}|_0 + 2\pi\beta\alpha\, W^{\mu}_{\nu}|_0. \,, \tag{3.78}$$

Moreover, the $(\chi\chi)$ component of the six-dimensional Einstein tensor evaluated at $\chi = 0$ is given in terms of brane quantities as

$$\frac{1}{M_6^4} T^{(B)\,\chi}_{\chi}\Big|_0 = -\frac{1}{2} R^{(4)}\Big|_0 - \frac{\alpha}{2}\left(R^{(4)\,2} - 4R^{(4)\,2}_{\mu\nu} + R^{(4)\,2}_{\mu\nu\kappa\lambda}\right)\Big|_0 - \frac{K'}{4}\Big|_0 - \frac{1}{8}K^{\nu}_{\mu}K^{\mu}_{\nu}|_0 + \frac{g'L'}{4gL}\Big|_0 + \frac{\nabla^{(4)}_{\mu}\partial^{\mu}L}{L}\Big|_0 + \frac{\Lambda_6}{M_6^4}\Big|_0. \tag{3.79}$$

In the above relation we have

$$\frac{K'}{4}\Big|_0 = 2\left(\frac{F''}{F} - \frac{F'^2}{F^2}\right)\Big|_0 \tag{3.80}$$

$$\frac{1}{8}K^{\nu}_{\mu}K^{\mu}_{\nu}|_0 = 2\frac{F'^2}{F^2}\Big|_0, \tag{3.81}$$

$$\frac{g'L'}{4gL}\Big|_0 = \frac{F'L'}{2FL}\Big|_0, \tag{3.82}$$

$$\frac{\nabla^{(4)}_{\mu}\partial^{\mu}L}{L}\Big|_0 = \frac{1}{F^2\,L}\left[2\dot{L}\left(\frac{A^2}{r} + A\dot{A}\right) + A^2\ddot{L}\right]\Big|_0, \tag{3.83}$$

and we can see that requiring $L(r,\chi=0)=0$ all terms are regular except (3.82) which has a $\frac{1}{\chi}$ contribution which is singular. This can only be avoided if $F'|_0 = 0$, thus $\alpha = \frac{L_4^2}{12}$, i.e., β constant. Another way to make the relation (3.79) regular is to take the pure Gauss-Bonnet case, where we do not take under consideration the induced gravity term in the action. In this case the bulk solutions are the same and we do not have the contributions (3.80)-(3.83) in (3.79) which becomes as (3.57). Then for both cases 1 and 2 if we want to match the $T^{(B)\,\chi}_{\chi}$ component of the bulk solution with the one derived in (3.79) we must have the relation $\alpha = \frac{\lambda^2}{12}$ which gives the constant β case. Thus, the relation (3.79) between bulk and brane quantities in order to be regular in the vicinity of the conical singularity requires the deficit angle to be constant.

4 The rôle of the Gauss-Bonnet Term

It is known that from a Ricci flat (D-1)-dimensional solution a D-dimensional solution can be generated which satisfies the Ricci flat D-dimensional Einstein equations [43]. This procedure can also be applied if there is a D-dimensional negative cosmological constant in the bulk. This result was used in [24] to construct the five-dimensional black string in codimension-1 branes.

If there is a Gauss-Bonnet term in the bulk there is a drastic change in this result [44, 45]. In the five-dimensional case consistency of the four-dimensional Einstein equations forces

the four-dimensional Gauss-Bonnet term projected on the brane to be constant [44] [1]. This implies that there could not exist black string solutions of the type in [24] with a Gauss-Bonnet term in the bulk in codimension-1 braneworlds.

In codimension-2 braneworlds there is a relation connecting the Gauss-Bonnet term projected on the brane with the components of the bulk energy-momentum tensor corresponding to the extra dimensions [8]. In six dimensions it reads

$$-\frac{1}{2}\,R^{(4)}|_0-\frac{1}{2}\alpha\left(R^{(4)\,2}-4R^{(4)\,2}_{\mu\nu}+R^{(4)\,2}_{\mu\nu\kappa\lambda}\right)\Big|_0=\frac{1}{M_6^4}\,T^{(B)\,\chi}_{\chi}|_0-\frac{\Lambda_6}{M_6^4}|_0\,. \tag{4.1}$$

All bulk solutions have to satisfy this relation which acts as a consistency relation. In spite of the fact that in four dimensions the Gauss-Bonnet term is a topological invariant, when it is projected on the brane, it leaves its traces through this relation. For the Schwarzschild-AdS solution of the form (3.18) the square of the Riemann tensor reads

$$R^2_{\mu\nu\kappa\lambda}=\frac{192\zeta^2 e^{\frac{4\chi}{L_4}}}{(1+e^{\frac{2\chi}{L_4}})^4 r^6}+\frac{60}{L_4^4}\,, \tag{4.2}$$

while the Ricci scalar and Ricci tensor are constants. Therefore, for the relation (4.1) to be satisfied the bulk energy-momentum tensor $T^{(B)\,\chi}_{\chi}|_0$ has to scale as $1/r^6$ with the right coefficients. This is actually what happens considering the result (3.24). Moreover, it is easy to verify that relation (4.1) is satisfied substituting the relevant quantities. Thus, the presence of the Gauss-Bonnet term in the bulk, which acts as a source term because of its divergenceless nature, dictates the form of matter that must be introduced in the bulk in order to sustain a black hole on the brane[2].

For the physically, most interesting solutions of the Schwarzschild-AdS black hole on the brane, we found that there must be non-trivial matter in the extra two dimensions given by (3.24). These components of the energy-momentum tensor depend on the radial distance on the brane r and on one of the extra dimensions χ through the warp function $F(\chi)$. Therefore, if we go far away from the brane (large χ) because of the form of the warp function (see Table 2) the energy momentum tensor coming from the bulk decouples. This means that on the brane we have standard four-dimensional gravity without any corrections from the bulk. On the contrary, near the brane the $1/r^6$ term dominates (the warp function goes to a constant) giving a strong modification of the four-dimensional gravity on the brane.

In five-dimensions a similar relation to (4.1) holds. Then, if we use the BTZ solution of Table 1 of section 2, the corresponding relation in five-dimensions is automatically satisfied. The reason is that the BTZ black hole does not have an $r=0$ curvature singularity [48] and, therefore, all the curvature invariants appearing in the relation are constants. Also the BTZ solution does not require matter in the bulk [39]. Thus, the corresponding relation to (4.1) in five dimensions is trivially satisfied, allowing the existence of a black string-like solution in five-dimensional braneworlds of codimensionality two.

[1]A similar relation obtained in [44] involving the Gauss-Bonnet term was presented in [46] in a different context.

[2]Black hole solutions in codimension-2 braneworlds were also recently discussed in [47].

The situation is more subtle for the sort distance BTZ-corrected solution of section 2. This black hole has $1/r$ curvature singularity giving, therefore, a non-constant Krestschmann scalar proportional to $1/r^6$. This implies that for the relation to hold the combination of the three-dimensional squared Ricci scalar and the squared Ricci tensor should also be proportional to $1/r^6$ with the appropriate coefficients. These curvature invariants can be obtained solving the three-dimensional Einstein equations on the brane (2.8). In order to get a non-trivial solution matter should be introduced on the brane, and this is actually what happens as it was shown in [39] (see relation (2.15)).

In the five-dimensional case we have found that the matter necessary to sustain the BTZ-corrected black hole solution on the brane is provided by a scalar field conformally coupled to the BTZ black hole. In six dimensions it is not clear to what system the "holographic matter" necessary to sustain the Schwarzschild-AdS black hole on the brane, corresponds. Considering the similarities between the five and six-dimensional cases it might correspond also to a scalar field coupled to the six-dimensional gravitational action.

5 Conclusions

We discussed black hole localization on an infinitely thin 3-brane of codimension-2 and its extension into a six-dimensional AdS bulk. To have a four-dimensional gravity on the brane we introduced a six-dimensional Gauss-Bonnet term in the bulk and an induced gravity term on the brane. We showed that a Schwarzschild-AdS black hole can be localized on the brane which is extended into the bulk with a warp function. Consistency of the six-dimensional bulk equations requires a fine-tuned relation between the Gauss-Bonnet coupling constant and the length of the six-dimensional AdS space. The use of this fine-tuning gives to the non-singular horizon the shape of a throat up to the horizon of the AdS space with no other curvature singularities except the Schwarzschild string-like singularity.

If the deficit angle is constant, independent of the radial coordinate of the brane, there is a consistency relation between the Gauss-Bonnet term projected on the brane and the energy-momentum tensor of the two extra dimensions. This relation for the Schwarzschild-AdS black hole solution on the brane requires the presence of a form of "holographic matter" in the extra dimensions which scales as $1/r^6$. This gives a strong modification of gravity at short distances while standard GR is obtained only at large distances.

If the deficit angle is variable, the effective four-dimensional Einstein equations on the brane acquire extra terms related to the projection of the Weyl tensor on the brane. Also, the constraint relation connecting the Gauss-Bonnet term projected on the brane and the bulk energy-momentum tensor is more involved, and in spite of the fact that the Schwarzschild-AdS black hole solution on the brane is still a solution of the bulk equations, it gives an inconsistency forcing the deficit angle to be constant.

The presence of the Gauss-Bonnet term is important in our considerations. It allows the existence of black string solutions in five-dimensions and in six dimensions it specifies the form of matter which is needed in the bulk in order to sustain a black hole on the brane. It would have been interesting to find out what modifications the gravitational action is needed in order to obtain bulk solutions without the need of matter in the extra dimensions.

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